Steady-state properties of a mean-field model of driven inelastic mixtures

Umberto Marini Bettolo Marconi[1,2] and Andrea Puglisi[3,4]

[1]*Dipartimento di Matematica e Fisica, Universitá di Camerino, 62032 Camerino, Italy*
[2]*Istituto Nazionale di Fisica della Materia, Unità di Camerino, Camerino, Italy*
[3]*Dipartimento di Fisica Universitá "La Sapienza," Piazzale Aldo Moro 2, 00198 Roma, Italy*
[4]*Istituto Nazionale di Fisica della Materia, Unità di Roma, Roma, Italy*

We investigate a Maxwell model of inelastic granular mixture under the influence of a stochastic driving and obtain its steady-state properties in the context of classical kinetic theory. The model is studied analytically by computing the moments up to the eighth order and approximating the distributions by means of a Sonine polynomial expansion method. The main findings concern the existence of two different granular temperatures, one for each species, and the characterization of the distribution functions, whose tails are in general more populated than those of an elastic system. These analytical results are tested against Monte Carlo numerical simulations of the model and are in general in good agreement. The simulations, however, reveal the presence of pronounced non-Gaussian tails in the case of an infinite temperature bath, which are not well reproduced by the Sonine method.

I. INTRODUCTION

Granular materials, a term coined to classify assemblies of macroscopic dissipative objects, are ubiquitous in nature and play a major role in many industrial and technological processes.

Interestingly, when a rarefied granular system is vibrated some of its properties are similar to those of molecular fluids, while others are unique and have no counterpart in ordinary fluids [1]. A spectacular manifestation of this difference can be observed in a driven mixture of granular particles: if one measures the average kinetic energy per particle, proportional to the so called granular temperature, one finds the surprising result that each species reaches a different value. Such a feature, observed in a recent experiment [2], is in sharp contrast with the experience with other states of matter. At a more fundamental level such a behavior is in conflict with the zeroth law of thermodynamics. This principle states that when two systems globally isolated are brought into thermal contact they exchange energy until they reach a stationary state of mutual equilibrium, characterized by the same value of their temperatures. A corollary to such a principle is the statement that the thermal equilibrium between systems A and B, i.e., $T_A = T_B$, and between A and C ($T_A = T_C$) implies the thermal equilibrium between B and C [3].

On the contrary, when two granular system A and B subject to an external driving force exchange energy they may reach in general a mutual equilibrium, characterized by two constant but different granular temperatures T_A and T_B. In addition even when A and C are in equilibrium at the same temperature T_C and B and C also have the temperature T_C one cannot conclude that A and B would be in mutual equilibrium at the same temperature. In other words, one of the most useful properties of the temperature, i.e., the independence from the thermal substance is lost when one deals with granular materials.

In the present paper we shall investigate the properties of a simple model of two-component granular mixture. The mo-tivation of our study relies on the fact that granular materials being mesoscopic objects are often constituted by assemblies of grains of different sizes and/or different physical and mechanical properties. The study of granular mixtures has attracted so far the attention both of theoreticians [4] and of experimentalists [2]. In particular Garzó and Dufty have studied the evolution of a mixture of inelastic hard spheres in the absence of external driving forces, a process termed free cooling because it is associated with a decrease of the average kinetic energy of the system, i.e., of its granular temperature. During the cooling of a mixture, which can be homogeneous or not, according to the presence of spatial density and velocity gradients, each species may have different granular temperatures, although these may result asymptotically proportional, i.e., they might decrease at the same rate.

Menon and Feitosa instead studied several mixtures of vibrated inelastic grains and reported their failure to reach the same granular temperature.

In the present paper our interest will be concentrated on the statistically stationary state obtained by applying an energy feeding mechanism represented by a stochastic driving force.

The most widely used model of granular materials is, perhaps, the inelastic hard sphere model (IHS) [5], which assumes the grains to be rigid and the collisions to be binary, instantaneous, and momentum conserving. The dissipative nature of the collisions is accounted for by values less than 1 of the so called restitution coefficient r. Even such an idealized model represents a hard problem to the theorist and one has to rely on numerical methods, namely, molecular dynamics or event driven simulation, or to resort to suitable truncation schemes of the hierarchy of equations for the distribution functions. One of these schemes is represented by the Boltzmann equation based on the molecular chaos hypothesis, which allows to study the evolution of the one-particle distribution function. Its generalization to the two-component mixture of inelastic hard spheres has been recently considered by Garzó and Dufty.

66 011301-1

Our treatment will depart from previous studies, because we have chosen an even simpler approach based on the so called gases of inelastic (pseudo)-Maxwell molecules. The inelastic Maxwell model has recently been the object of vivid interest because it illustrates in a transparent fashion the non trivial role of inelasticity on the properties of the system. These gases are natural extensions to the inelastic case of the models of Maxwell molecules [6], where the collision rate is independent of the relative velocity of the particles. Such a feature greatly simplifies both the analytical structure of the Boltzmann equation and the numerical implementation of the algorithm simulating the gas dynamics. Although the constant collision rate is somehow unrealistic one may hope to be able to capture some salient features of granular mixtures, and in particular to reach a better understanding of their global behavior, because the model lends itself to analytical studies.

This type of approach to granular gases had recently a surge of activity since the work of Ben-Naim and Krapivsky [7,8], our group [9–11], Ernst and co-workers [12], and Cercignani and co-workers [13,14], and is providing a series of important results concerning the energy behavior and the anomalous velocity statistics of granular systems.

The paper is organized as follows: in Sec. II we define the model and deduce the associated Boltzmann equations governing the evolution of the velocity distribution functions of the two species and set up their moment expansion; in Sec. III we shall determine the exact values of the stationary granular temperatures; in Sec. IV we consider the moments up to the eighth order and compute the distribution functions by means of the so called Sonine expansion. In Sec. V we simulate on a computer the dynamics of the inelastic mixture and compute numerically the distribution functions and compare these with the theoretical predictions. Finally in Sec. VI we present our conclusions.

II. DEFINITION OF THE MODEL

In the following we shall consider a mixture of Maxwell inelastic molecules subject to a random external driving. Let us consider an assembly of N_1 particles of species 1 and N_2 particles of species 2. For the sake of simplicity we assume the velocities v_i^α, with $\alpha = 1,2$, to be scalar quantities. The two species may have different masses, m_1 and m_2 and/or constant restitution coefficients that depend on the nature of the colliding grains but not on their velocities, i.e., r_{11}, r_{22}, and $r_{12} = r_{21}$.

The mixture evolves according to the following set of stochastic equations:

$$m_i \frac{dv_i}{dt} = F_i + f_i + \xi_i(t), \tag{1}$$

where the total force acting on particle i is made of three contributions: the impulsive force F_i due to mutual collisions, the velocity dependent force $f_i = -\Gamma v_i$ due to the friction of the particles with the surroundings and the stochastic force ξ_i due to an external random drive. Since we are inter-

ested in the rapid granular flow regime, we model the collisions as instantaneous binary events, similar to those occurring in a hard-sphere system.

The presence of the frictional, velocity dependent term in addition to the random forcing [15], not only mimics the presence of friction of the particles with the container, but also is motivated by the idea of preventing the energy of a driven elastic system ($\gamma \rightarrow 1$), to increase indefinitely.

Let us observe that in the absence of collisions the velocity changes are described by the following Ornstein-Uhlenbeck process:

$$m_\alpha \partial_t v_i(t) = -\Gamma v_i(t) + \xi_i^\alpha(t), \tag{2}$$

where the stochastic acceleration term is assumed to have a white spectrum with zero mean,

$$\langle \xi_i^\alpha(t) \rangle = 0 \tag{3}$$

and variance

$$\langle \xi_i^\alpha(t) \xi_j^\beta(t') \rangle = 2D \delta_{\alpha,\beta} \delta_{i,j} \delta(t-t') \tag{4}$$

By redefining the bath constants $\Gamma_\alpha = \Gamma/m_\alpha$ and $D_\alpha = D/m_\alpha^2$, it is straightforward to obtain the probability distributions of the velocity of each species, $P_\alpha(v,t)$. In fact, the Fokker-Planck equations associated with the process (2),

$$\partial_t P_\alpha(v,t) = \Gamma_\alpha \partial_v [v P_\alpha(v,t)] + D_\alpha \partial_v^2 P_\alpha(v,t) \tag{5}$$

possess the following stationary distribution functions:

$$P_\alpha(v) = \sqrt{\frac{m_\alpha}{2\pi T_b}} e^{-(m_\alpha v^2)/(2T_b)}, \tag{6}$$

where $T_b = D/\Gamma$ represents the temperature of the heath bath, that we fix to be the same for the two species (see also Appendix B).

In order to represent the effect of the collisions on the evolution of the system we assume that the velocities change instantaneously according to the rules

$$v_i'^\alpha = v_i^\alpha - [1 + r_{\alpha\beta}] \frac{m_\beta}{m_\alpha + m_\beta} (v_i^\alpha - v_j^\beta), \tag{7a}$$

$$v_j'^\beta = v_j^\beta + [1 + r_{\alpha\beta}] \frac{m_\alpha}{m_\alpha + m_\beta} (v_i^\alpha - v_j^\beta), \tag{7b}$$

where the primed quantities are the postcollisional velocities and the primed are the velocities before the collisions. A finite fraction of the kinetic energy of each pair is dissipated during a collision. Between collisions the velocities perform a random walk due to the action of the heat bath. In such a model the typical time τ_c between particle-particle collision is an adjustable parameter and is assumed to be small compared to the heat bath relaxation times, which are $\tau_{b1} = m1/\Gamma = 1/\Gamma_1$ and $\tau_{b2} = m_2/\Gamma = 1/\Gamma_2$. When $\Gamma \rightarrow 0$ we must also take $D \rightarrow 0$ in performing the elastic limit, otherwise the kinetic energy would diverge asymptotically. In fact in Sec.

V we shall discuss the situation of inelastic particles with vanishing friction, a case already considered in Refs. [7,13].

The evolution equations for the probability densities of finding particles of species α with velocity v at time t for the system subject both to external forcing and to collisions are simply obtained by adding the two effects,

$$\partial_t P_\alpha(v,t) = \Gamma_\alpha \partial_v [v P_\alpha(v,t)] + D_\alpha \partial_v^2 P_\alpha(v,t)$$
$$+ \frac{1}{\tau_c} Q_\alpha(P_1, P_2), \tag{8}$$

where the collision integrals Q_α consist of a negative loss term and a positive gain term respectively,

$$Q_1(P_1, P_2) = -P_1(v,t)$$

$$+ \frac{2p}{1+r_{11}} \int du P_1(u,t) P_1 \left(\frac{2v - (1-r_{11})u}{1+r_{11}}, t \right)$$

$$+ \frac{(1-p)}{1+r_{12}} \frac{m_1 + m_2}{m_2} \int du P_1(u,t) P_2$$

$$\times \left(\frac{\frac{m_1+m_2}{m_2} v - \left(\frac{m_1}{m_2} - r_{12} \right) u}{1+r_{12}}, t \right), \tag{9a}$$

$$Q_2(P_1, P_2) = -P_2(v,t) + \frac{2(1-p)}{1+r_{22}} \int du P_2(u,t) P_2$$

$$\times \left(\frac{2v - (1-r_{22})u}{1+r_{22}}, t \right)$$

$$+ \frac{p}{1+r_{12}} \frac{m_1 + m_2}{m_1} \int du P_2(u,t) P_1$$

$$\times \left(\frac{\frac{m_1+m_2}{m_1} v - \left(\frac{m_2}{m_1} - r_{12} \right) u}{1+r_{12}}, t \right), \tag{9b}$$

where $p = N_1/(N_1+N_2)$. In writing Eqs. (9) we have assumed that the collisions occur instantaneously and that collisions involving more than two particles simultaneously can be disregarded. Moreover, all pairs are allowed to exchange impulse regardless of their mutual separation, in the spirit of a mean-field model. In order to proceed further it is convenient to take Fourier transforms of Eqs. (9) and employ the method of characteristic functions [16] defined as

$$\hat{P}_\alpha(k,t) = \int_{-\infty}^{\infty} dv\, e^{ikv} P_\alpha(v,t) \tag{10}$$

The resulting equations read

$$\partial_t \hat{P}_1(k,t) = -D_1 k^2 \hat{P}_1(k,t) - \Gamma_1 k \partial_k \hat{P}_1(k,t)$$

$$- \frac{1}{\tau_c} \{ \hat{P}_1(k,t) - p\hat{P}_1(\gamma_{11}k,t)\hat{P}_1[(1-\gamma_{11})k,t]$$

$$- (1-p)\hat{P}_1(\tilde{\gamma}_{12}k,t)\hat{P}_2[(1-\tilde{\gamma}_{12})k,t]\}, \tag{11a}$$

$$\partial_t \hat{P}_2(k,t) = -D_2 k^2 \hat{P}_2(k,t) - \Gamma_2 k \partial_k \hat{P}_2(k,t) - \frac{1}{\tau_c} [\hat{P}_2(k,t)$$

$$- (1-p)\hat{P}_2(\gamma_{22}k,t)\hat{P}_2[(1-\gamma_{22})k,t]$$

$$- p\hat{P}_2(\tilde{\gamma}_{21}k,t)\hat{P}_1[(1-\tilde{\gamma}_{21})k,t]] \tag{11b}$$

with

$$\gamma_{\alpha\beta} = \frac{1-r_{\alpha\beta}}{2}, \tag{12a}$$

$$\tilde{\gamma}_{12} = \left[1 - \frac{2}{1+\zeta}(1-\gamma_{12}) \right], \tag{12b}$$

$$\tilde{\gamma}_{21} = \left[1 - \frac{2}{1+\zeta^{-1}}(1-\gamma_{12}) \right], \tag{12c}$$

with $\zeta = m_1/m_2$

The mathematical structure of Eqs. (11) is particularly simple and in fact there exists a standard method of solution. It consists in expanding the Fourier transform $\hat{P}_\alpha(k,t)$ of the distributions $P_\alpha(v,t)$ in a Taylor series around the origin $k=0$,

$$\hat{P}_\alpha(k,t) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n^\alpha(t) \tag{13}$$

and substituting Eq. (13) into Eqs. (11).

Equating like powers of k we obtain a hierarchy of equations for the $\mu_n^\alpha(t)$ which can be solved by a straightforward iterative method. At this stage one can appreciate the mathematical convenience of the Maxwell model. In fact, the coefficients of the Taylor series represent the moments of the velocity distributions,

$$\mu_n^\alpha(t) = \int_{-\infty}^{\infty} dv\, v^n P_\alpha(v,t). \tag{14}$$

Since the evaluation of the moments of a given order requires only the knowledge of the moments of lower order one can proceed without excessive difficulty to any desired order. In practice, we carried on our calculation up to the eighth moment, assuming that the initial distributions were even, so that the odd moments vanish. In order to render the reading of the paper more expeditious we shall report the equations determining the stationary value of the moments in the Appendix.

III. TWO-TEMPERATURE BEHAVIOR

In order to determine the granular temperatures we equate the coefficients of order k^2 in Eqs. (11) and obtain the governing equations for the second moments,

$$\tau_c \partial_t \mu_2^{(1)} = \{p[2\gamma_{11}(\gamma_{11}-1)] - (1-p)(1-\tilde{\gamma}_{12}^2)$$
$$-2\tau_c \Gamma_1\}\mu_2^{(1)} + (1-p)(1-\tilde{\gamma}_{12})^2 \mu_2^{(2)} + 2\tau_c D_1,$$

$$\text{(15a)}$$

$$\tau_c \partial_t \mu_2^{(2)} = \{(1-p)[2\gamma_{22}(\gamma_{22}-1)] - p(1-\tilde{\gamma}_{21}^2)$$
$$-2\tau_c \Gamma_2\}\mu_2^{(2)} + p(1-\tilde{\gamma}_{21})^2 \mu_1^{(2)} + 2\tau_c D_2$$

$$\text{(15b)}$$

The right-hand side of Eqs. (15) represent the balance between the energy dissipation due to inelastic collisions and friction and the energy input due to the bath. We define the global and the partial granular temperatures respectively as $T_g = pT_1 + (1-p)T_2$ and $T_\alpha = \frac{1}{2}m_\alpha \langle v_\alpha^2 \rangle$, where the average is performed over the noise ξ_i. Since the energy dissipation and the energy supply mechanisms compete, the system under the influence of a stochastic white noise driving achieves asymptotically a statistical steady state. Notice that Eqs. (15) feature only the second moments of the velocity distributions, so that the solution is straightforward. Such a state of affairs should be contrasted with the analog problem of determining the partial temperatures in Boltzmann models [4].

Let us start analyzing the behavior of the Maxwell gas in the one component limit $p \rightarrow 1$. The granular temperature approaches its stationary value T_1 exponentially,

$$T_1(t) = T_1(0)e^{-2t/\tau} + T_1(\infty)[1 - e^{-2t/\tau}], \quad \text{(16)}$$

where the constant τ represents a combination of the two characteristic times of the process given by

$$\frac{1}{\tau} = \frac{\gamma_{11}(1-\gamma_{11})}{\tau_c} + \frac{\Gamma}{m_1}, \quad \text{(17)}$$

which shows that T_b is an upper bound to the granular temperature. We also obtain a simple relation between the temperature of the bath and the granular temperature T_1,

$$T_{1\infty} = m_1 \frac{D_1}{\Gamma_1 - \frac{\gamma_{11}(\gamma_{11}-1)}{\tau_c}} \leqslant \frac{D}{\Gamma} = T_b. \quad \text{(18)}$$

On the other hand, when the two components are not identical Eqs. (15) show that the equilibrium macrostate is specified by two different partial granular temperatures, both proportional to the heath-bath temperature. This means that the ratio of the two temperatures is independent from the driving intensity D. This is also seen from the formula

$$\frac{T_1}{T_2} = \frac{1}{\zeta} \frac{(1-p)[2\gamma_{22}(1-\gamma_{22})] + p(1-\tilde{\gamma}_{21}^2) + 2\frac{\tau_c}{\tau_{b2}} + (1-p)\zeta^2(1-\tilde{\gamma}_{12})^2}{p[2\gamma_{11}(1-\gamma_{11})] + (1-p)(1-\tilde{\gamma}_{12}^2) + 2\frac{\tau_c}{\tau_{b1}} + p\zeta^{-2}(1-\tilde{\gamma}_{21})^2}. \quad \text{(19)}$$

Formula in Eq. (19) illustrates the two-temperature behavior of an inelastic mixture subject to external driving. Notice that the temperature ratio in the driven case is different from the corresponding quantity in the cooling undriven case, for the same model system. In the undriven case we found that the homogeneous cooling state was characterized by two different exponentially decreasing temperatures, but whose ratio was constant. However, no simple relation exists between the ratio relative to the two cases, on account of the fact that the energy exchanges involved are rather different. In the presence of a heath bath the inelastic mixture displays the two-temperature behavior already reported in the free cooling case [4,17] and in experiments [2]. This feature seems to be a general property of inelastic systems. In Fig. 1 we display the temperature ratio as a function of the mass ratio ζ for two different values of the inelasticity. Notice that the temperature of the heavier component is lower than the one of the lighter species. Finally we notice that in the cooling case the temperature ratio was a increasing function of ζ, while in the present model it is a decreasing function of the same variable. Such an effect depends crucially on the chosen form of the heat bath. In Appendix B we discuss another choice of the heat-bath parameters that make T_1/T_2 an increasing function of ζ.

In Fig. 2 the ratio of the temperatures of the two species is plotted in the case of an asymmetry in the restitution coefficients parametrized by the form: $r_{22} = r_{11} - x$ with $r_{12} = (r_{22} + r_{11})/2$, for $p = 1/2$, identical masses and three different values of the coefficient r_{11} as shown in figure. In Fig. 3 we display the same quantity as a function of concentration p for different choices of the other parameters. One sees that the
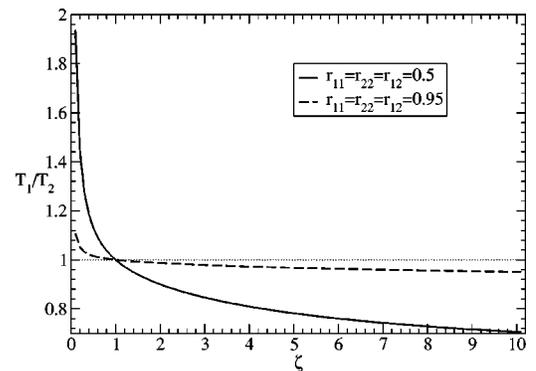

FIG. 1. Temperature ratio as a function of the mass ratio for different choices of the inelasticity parameter, for $p = 0.5$, and $r_{11} = r_{22} = r_{12} = r = 0.95$ (solid line) and $r = 0.5$ (dashed line).
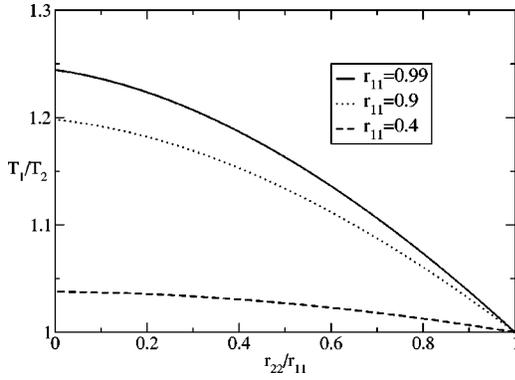
FIG. 2. Temperature ratio as a function of the asymmetry r_{22}/r_{11} for $p=0.5$, $\zeta=1$, and different choices of the inelasticity parameter r_{11}, from top to bottom this is respectively 0.99, 0.9, 0.4 (line).

variation of T_1/T_2 is much smaller than the corresponding variation with respect to the mass asymmetry shown in Fig. 1 in agreement with the experimental observation of Ref. [2]. It seems reasonable to conclude that the mass asymmetry is the larger source of temperature difference between the two components.

IV. VELOCITY DISTRIBUTION FUNCTIONS

An interesting aspect of granular systems concerns the nature of the single particle velocity distributions. The inelasticity, in fact, causes marked departures of $P_\alpha(v,t)$ from the Gaussian form which characterizes gases at thermal equilibrium. In undriven gases these deviations are particularly pronounced and one observes inverse power law high-velocity tails both in gases of pseudo-Maxwell molecules [17] and in IHS [18]. In the driven case, i.e., in systems subject to Gaussian white noise forcing [similar to that represented by Eq. (2) with $\Gamma=0$] exponential tails of the form $\exp(-v^{3/2})$ have been predicted theoretically in inelastic hard-sphere models [18] and tested by direct simulation Monte Carlo of the Enskog-Boltzmann equation [19]. Have these non-Gaussian tails a counterpart in Maxwell models? Ben-Naim and Krapivsky [7] on the basis of a resummation of the moment expansion concluded that the scalar Maxwell
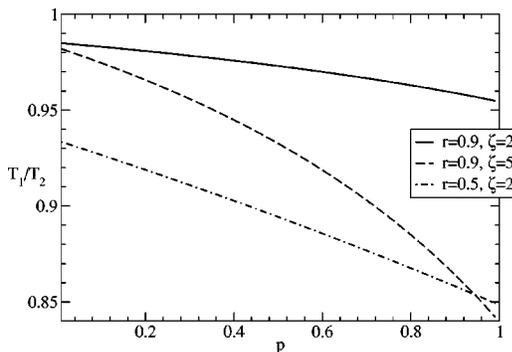

FIG. 3. Temperature ratio as a function of the molar fraction p for different choices of the inelasticity parameter $r=r_{11}=r_{22}$ $=r_{12}$, and of the mass ration ζ.

models with vanishing viscosity ($\Gamma=0$) should display Gaussian-like tails. However, this prediction is in contrast with the argument, employed by Ernst and van Noije in the case of IHS, which consists in estimating the tails of the distribution by linearizing the master equation (9) by neglecting the gain term. This assumption simplifies the analysis and allows us to reach the conclusion that the velocity distribution for large v should vanish as

$$\lim_{v\to\infty} P(v)\propto\exp(-v/v_0), \qquad (20)$$

with $v_0^2=D\tau_c$. Clearly such a result is in sharp contrast with the result of Ref. [7] and seems to indicate that the Sonine expansion does not reproduce faithfully the high-velocity tails in the case of Maxwell models with vanishing viscosity. The test of the limit (20) will be shown in Sec. VI, where we illustrate the results of our numerical simulations.

On the other hand, the same kind of asymptotic analysis sketched above, allows us to conclude that the presence of a viscous damping is the redeeming feature that renders convergent the Sonine expansion and the associated Gaussian tails. In fact, with a finite value of Γ the asymptotic solution is of the form

$$\lim_{v\to\infty} P(v)\propto\exp(-Cv^2). \qquad (21)$$

We shall test such a prediction in the remaining part of this section and study the velocity distributions of the individual species when $\Gamma\neq0$ by constructing the solution to the master equation using the Sonine polynomial expansion method, one of the traditional approaches to the solution of the Boltzmann equation [20].

We shall also investigate whether the two partial distributions can be cast into the same functional form upon rescaling the velocities with respect to the partial granular thermal velocity, in other words if it is possible to have a data collapse for the two distributions.

We shall first obtain the steady-state values of the first eight moments as illustrated in the Appendix and then compute the approximate form of the distribution functions by assuming that these are Gaussians multiplied by a linear combination of Sonine polynomials.

Let us begin by writing the following Sonine expansion of the distribution functions:

$$f_\alpha(c)=\frac{1}{\sqrt{\pi}}e^{-c^2}\left[1+\sum_{n=1}^{\infty}a_n^\alpha S_n(c^2)\right], \qquad (22)$$

where $f_\alpha(c)$ is the rescaled distribution defined by

$$f_\alpha(c)=\sqrt{2\mu_2^\alpha}P_\alpha(v) \qquad (23)$$

and $c^2=v^2/2\mu_2^\alpha$. The expansion gives the distributions in terms of the coefficients a_n^α of the Sonine polynomials $S_n(c^2)$. In practice, one approximates the series (22) with a finite number of terms. Since the leading term is the Max-
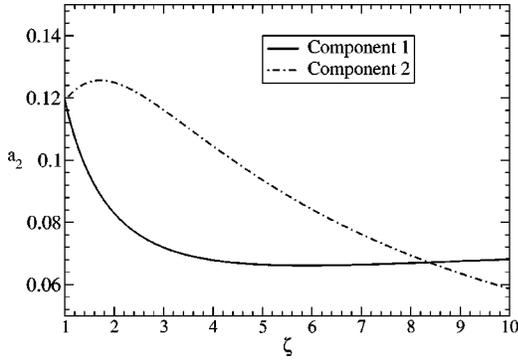
FIG. 4. Second coefficient of the Sonine expansion a_2, for each component as a function of the mass ratio ζ, $m_1=1$ and for $p=0.5$ and $r_{11}=r_{22}=r_{12}=0.5$. We have also fixed $1/\Gamma \equiv \tau_{b1}=200$ and $\tau_c=25$.

wellian, the closer the system is to the elastic limit, the less terms suffice to describe the state. The expression of the first polynomials is

$$S_0(c^2)=1, \tag{24a}$$

$$S_1(c^2)=\frac{1}{2}-c^2, \tag{24b}$$

$$S_2(c^2)=\frac{3}{8}-\frac{3}{2}c^2+\frac{1}{2}c^4, \tag{24c}$$

$$S_3(c^2)=\frac{5}{16}-\frac{15}{8}c^2+\frac{5}{4}c^4-\frac{1}{6}c^6, \tag{24d}$$

$$S_4(c^2)=\frac{35}{128}-\frac{35}{16}c^2+\frac{35}{16}c^4-\frac{7}{12}c^6+\frac{1}{24}c^8. \tag{24e}$$

In order to obtain the first m values a_m^α, we need to compute the rescaled moments $\langle c^n \rangle_\alpha$ of the distribution functions up to order $2m$. These moments are evaluated in Appendix by means of a straightforward iterative method. At the end, knowing the rescaled moments, one obtains the following relation for the coefficients:
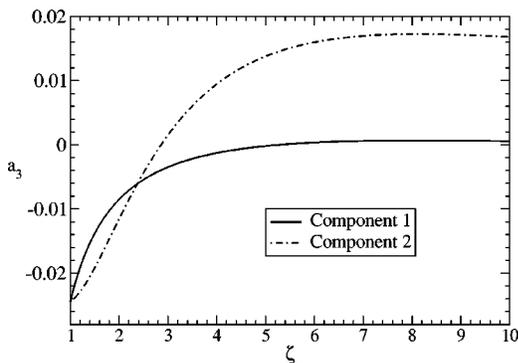

FIG. 5. Third coefficient of the Sonine expansion a_3, for each component as a function of the mass ratio ζ. The remaining parameters are as in Fig. 4.
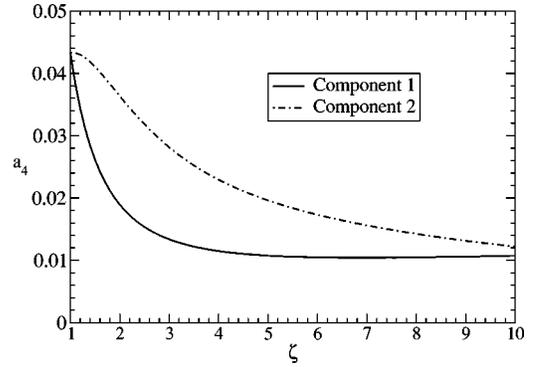
$$a_n^\alpha=\frac{\langle S_n(c^2) \rangle_\alpha}{\mathcal{N}_n}. \tag{25}$$

Equation (25) can be proved by imposing the consistency condition

$$\langle c^n \rangle_\alpha=\int_{-\infty}^{\infty}dc\,c^n f_\alpha(c) \tag{26}$$

in conjunction with the orthogonality property of the Sonine polynomials,

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-c^2}S_n(c^2)S_m(c^2)=\mathcal{N}_n\delta_{m,n}, \tag{27}$$

where $\mathcal{N}_n$ is a normalization constant. Notice that in order to obtain our results we have not assumed weak inelasticity, therefore these hold for any value of the restitution coefficients.

The coefficients a_m^α up to the fourth order in terms of the rescaled moments read

$$a_1^\alpha=0, \tag{28a}$$

$$a_2^\alpha=\left[1-4\langle c^2 \rangle_\alpha+\frac{4}{3}\langle c^4 \rangle_\alpha\right], \tag{28b}$$

$$a_3^\alpha=\left[1-6\langle c^2 \rangle_\alpha+4\langle c^4 \rangle_\alpha-\frac{8}{15}\langle c^6 \rangle_\alpha\right], \tag{28c}$$

$$a_4^\alpha=\left[1-8\langle c^2 \rangle_\alpha+8\langle c^4 \rangle_\alpha-\frac{32}{15}\langle c^6 \rangle_\alpha+\frac{16}{105}\langle c^8 \rangle_\alpha\right]. \tag{28d}$$

In Figs. 4–6 we display the behavior of the Sonine coefficients for both components in the case of equal restitution coefficients $r=0.5$ as a function of the mass ratio.

In Figs. 7–9 we illustrate the variation of the Sonine coefficients for the two components as a function of the inelasticity for two different values of the mass ratio: $\zeta=1$ and $\zeta=2$. Notice that the coefficients are monotonic functions of the inelasticity as already noticed in pure systems [18].

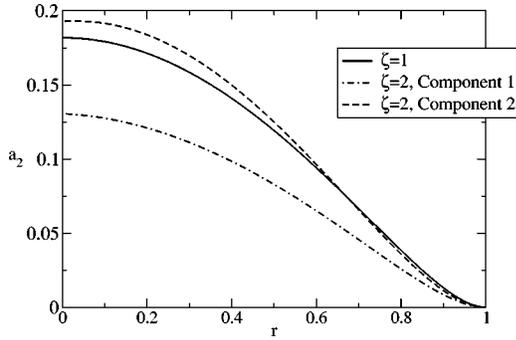In Fig. 10 we show the distribution functions for the

FIG. 6. Fourth coefficient of the Sonine expansion a_4, for each component as a function of the mass ratio ζ. The remaining parameters are as in Fig. 4.

FIG. 7. Second coefficient of the Sonine expansion a_2, for each component as a function of the inelasticity $r = r_{11} = r_{22} = r_{12}$, for $p = 0.5$ and $m_1 = 1$ and $\zeta = 2$ and $\zeta = 1$. The remaining parameters are as in Fig. 4.

FIG. 9. Fourth coefficient of the Sonine expansion a_4, for each component as a function of the inelasticity $r = r_{11} = r_{22} = r_{12}$, for $p = 0.5$ and $m_1 = 1$ and $\zeta = 2$ and $\zeta = 1$. The remaining parameters are as in Fig. 4.

heated system with nonvanishing viscosity. The tails become fatter with increasing order of the approximation, i.e., the high-energy tails are overpopulated. Moreover, one sees that when $p = 1/2$ and the restitution coefficients are all equal the species with the larger tails is the lighter. On the other hand, for a system with the same masses, but different restitution coefficients, the more elastic species displays the larger tails. We also show the numerical data obtained by simulating the the dynamics. The agreement is quite satisfactory and validates the approximation method employed.

On the other hand, it is also evident, that the two distributions fail to collapse one over the other after the rescaling of the velocities. This fact is consistent with the different values assumed by the coefficients a_n^1 and a_n^2. However, this effect is rather small and can be appreciated only by studying the high-velocity region of the distribution functions.

V. NUMERICAL SIMULATIONS

To investigate the validity of the previous results and in particular to test the convergence of the Sonine expansion in different situations we shall present in this section numerical results obtained by simulating an ensemble of N particles subject to a Gaussian forcing, viscous friction, and inelastic collisions.

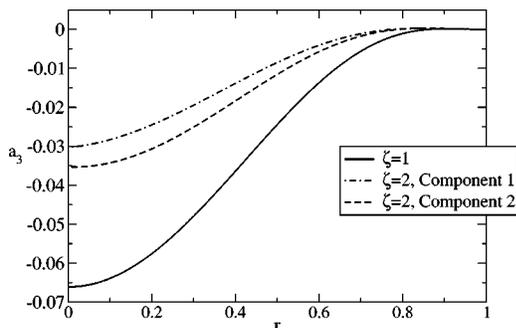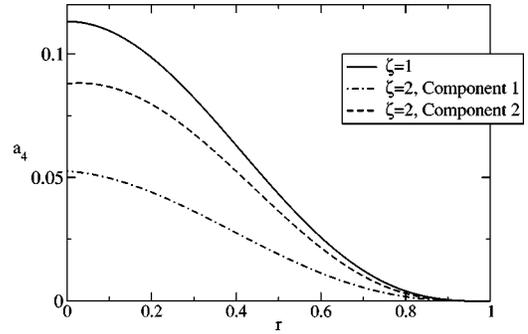The scheme consists of the following ingredients,

(i) Time is discretized, i.e., $t = n \times dt$.

(ii) Update all the velocities to simulate the random forcing and the viscous damping,

$$v_i^\alpha(t+dt) = v_i^\alpha(t) e^{-dt/\tau_b} + \sqrt{T_b(1 - e^{-(2dt/\tau_b)})} W(t), \quad (29)$$

where $W(t)$ is a normally distributed deviate with zero mean and unit variance.

(iii) Choose randomly $N(dt)/(2\tau_c)$ pairs of velocities and update each of them with the collision rule (7). In this way a mean collision time τ_c per particle is guaranteed.

(iv) Change the time counter n and restart from (ii).

In other words, at every step each particle experiences a Gaussian kick thus receiving energy from the bath, whereas it dissipates energy by collision and by damping. For example, by choosing $dt = 1$, $m_1/\Gamma = \tau_{b1} = 200$, and $\tau_c = 25$, we obtain that each particle in the average experiences 25 Gaussian kicks between two successive collisions and that the resulting average kinetic energy is stationary. In order to compare our numerical simulations with the theoretical predictions we fixed the temperature of the bath to be $T_b = 1$, i.e., chosen $D = \Gamma$. The results of such simulations are presented in Fig. 10 and show a very good agreement between the theory and the simulation.
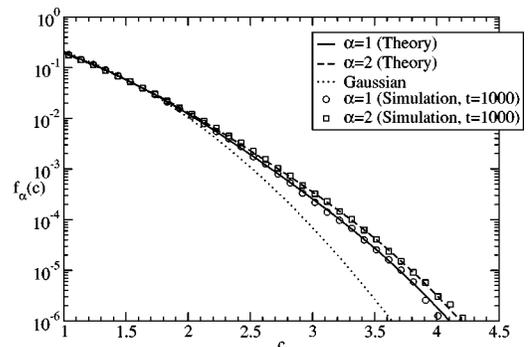

FIG. 8. Third coefficient of the Sonine expansion a_3, for each component as a function of the inelasticity $r = r_{11} = r_{22} = r_{12}$, for $p = 0.5$ and $m_1 = 1$ and $\zeta = 2$ and $\zeta = 1$. The remaining parameters are as in Fig. 4.

FIG. 10. Rescaled distribution functions for the two components for a model system characterized by $m_1 = 1$, $m_2 = 0.5$ and equal inelasticities $r = 0.5$, $T_b = 1$ and the remaining parameters as in Fig. 4.
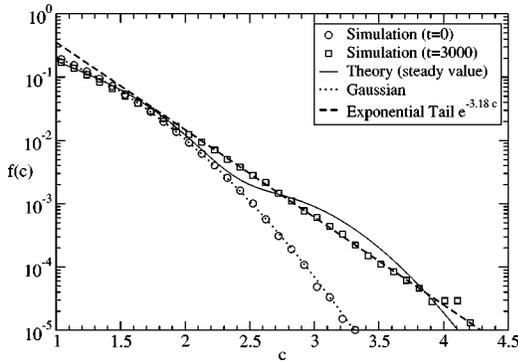
FIG. 11. Rescaled distribution functions for a one component system ($p=1$) with vanishing viscosity ($\Gamma=0$), $D=0.000\,8$, $r=0.5$, and $\tau_c=250$. We show the initial Gaussian distribution ($t=0$) and the asymptotic ($t=3000$) stationary distribution. Notice the presence of high-velocity tails. For the sake of comparison we report the theoretical estimate of the distribution obtained by means of the Sonine expansion.

On the contrary, the agreement between the Sonine expansion and the simulation is not completely satisfactory when we consider a system subject to a white noise acceleration, but without viscous friction, a driving proposed by some authors [21,22]. This can be considered as the limit $\Gamma \rightarrow 0$, $T_b \rightarrow \infty$, keeping constant $D=T_b\Gamma$, in the model defined by Eqs. (8): note that in this case the elastic limit $r \rightarrow 1$ cannot be performed without taking also the limit $D \rightarrow 0$ as discussed at the beginning, in order to avoid a divergence of kinetic energy. For the sake of simplicity, we simulated a one-component system ($p=1$ and $m=1$) with vanishing viscosity $\Gamma=0$, but $D=0.000\,8$, $\tau_c=250$, and $r=0.5$. Such a choice yields a granular temperature $T_g=16/15$, as predicted by our formula (18). Notice that in this case the heath-bath temperature diverges and the gas does not have a proper elastic limit, since all moments diverge when $\gamma \rightarrow 0$. We observed that the tails of the velocity distribution function are strongly non-Gaussian. These decay as a simple exponential as predicted by our simple analysis of the preceding section. In Fig. 11 we report our simulation results against the Sonine approximation. We observe that the theoretical estimate, in spite of incorporating the exact values of the first eight moments, deviates from the numerical data in the large velocity region. In particular the Sonine expansion can only give Gaussian tails, whereas the simulation indicates a slower exponential decay. The reason for such a discrepancy is to be ascribed to the slow convergence of the expansion when $\Gamma=0$.

VI. DISCUSSION AND CONCLUSIONS

To summarize we have studied the behavior of a model, which perhaps represent the simplest description of a driven inelastic gas mixture, namely, an assembly of two types of scalar pseudo-Maxwell molecules subject to a stochastic forcing. We have obtained the velocity distributions for arbitrary values of the inelasticity, of the composition and of the masses by solving the associated Boltzmann equation by means of a controlled approximation, the moment expansion.

The distributions were obtained by computing exactly the moments up to the eighth order and then imposing that the corrections to the Maxwell distribution stemming from the inelasticity are Gaussians multiplied by a linear combination of Sonine polynomials with amplitudes determined self-consistently.

The model predicts a steady two-temperature behavior that is in qualitative agreement with existing experimental results. The granular temperatures can be obtained by very simple algebraic manipulations for arbitrary values of the control parameters.

By numerical simulations we demonstrated that the velocity distributions are well described by our series representation in the case of systems in contact with a bath at finite temperature T_b, whilst the series expansion breaks down in the case of systems in contact with bath at infinite temperature, i.e., with zero viscosity.

What can be learned from such a simple model of granular mixture? Besides obtaining a global picture of the behavior of the system with a minimal numerical effort both in the cooling and in the driven case, the model displays the novel feature of two different distribution functions, which remain different even after rescaling by the associated partial granular temperatures. Of course the detailed form of the probability velocity distributions are strictly model dependent, i.e., depend on the assumption of a constant collision rate inherent in Maxwell models. The vectorial character of the velocities could be included at the cost of a moderate additional effort. A more interesting and difficult problem would be that of including in the mixture case a collision frequency proportional to an appropriate function of the kinetic granular temperatures, generalizing the work of Cercignani [13].

Finally we might ask the general question of the meaning of granular temperature. Our findings seem to indicate that it is still the main statistical indicator of the model granular system we studied. However, with respect to the temperature of a perfectly elastic system it fails to satisfy a very basic requirement which is known as the zeroth principle of thermodynamics.

APPENDIX A

In the present Appendix we sketch the derivation of the various moments of the distribution functions. By equating the equal powers of k in the master equation (11) we obtain a set of linear coupled equations for the moments. The method of solution is iterative, because the higher moments depend on the lower moments. Thus, for instance, to evaluate the lowest-order moments of the distribution functions we must solve the following equations for the steady-state value of the fourth moments:

$$(d_{11}^{(4)} - 4\Gamma_1\tau_c)\mu_4^{(1)} + d_{12}^{(4)}\mu_4^{(2)} + a_{11}(\mu_2^{(1)})^2 + a_{12}\mu_2^{(1)}\mu_2^{(2)}$$
$$+ 12D_1\tau_c\mu_2^{(1)} = 0, \tag{A1a}$$

$$(d_{22}^{(4)} - 4\Gamma_2\tau_c)\mu_4^{(2)} + d_{21}^{(4)}\mu_4^{(1)} + a_{22}(\mu_2^{(2)})^2 + a_{21}\mu_2^{(1)}\mu_2^{(2)}$$
$$+ 12D_2\tau_c\mu_2^{(2)} = 0. \tag{A1b}$$

In turn, the sixth moments are obtained by solving

$$(d_{11}^{(6)} - 6\Gamma_1\tau_c)\mu_6^{(1)} + d_{12}^{(6)}\mu_6^{(2)} + b_{11}\mu_2^{(1)}\mu_4^{(1)} + b_{12}\mu_2^{(1)}\mu_4^{(2)}$$
$$+ b_{12}'\mu_2^{(2)}\mu_4^{(1)} + 30D_1\tau_c\mu_4^{(1)} = 0, \tag{A2a}$$

$$(d_{22}^{(6)} - 6\Gamma_2\tau_c)\mu_6^{(2)} + d_{21}^{(6)}\mu_6^{(1)} + b_{22}\mu_2^{(2)}\mu_4^{(2)} + b_{21}\mu_2^{(1)}\mu_4^{(2)}$$
$$+ b_{21}'\mu_2^{(2)}\mu_4^{(1)} + 30D_2\tau_c\mu_4^{(2)} = 0. \tag{A2b}$$

Finally the eight moments are the solutions of

$$(d_{11}^{(8)} - 8\Gamma_1\tau_c)\mu_8^{(1)} + d_{12}^{(8)}\mu_8^{(2)} + c_{11}\mu_2^{(1)}\mu_6^{(1)} + c_{11}'(\mu_4^{(1)})^2$$
$$+ c_{12}\mu_2^{(1)}\mu_6^{(2)} + c_{12}'\mu_4^{(1)}\mu_4^{(2)} + c_{12}''\mu_6^{(1)}\mu_2^{(2)}$$
$$+ 56D_1\tau_c\mu_6^{(1)} = 0, \tag{A3a}$$

$$(d_{22}^{(8)} - 8\Gamma_2\tau_c)\mu_8^{(2)} + d_{21}^{(8)}\mu_8^{(1)} + c_{22}\mu_2^{(2)}\mu_6^{(2)} + c_{22}'(\mu_4^{(2)})^2$$
$$+ c_{21}\mu_2^{(2)}\mu_6^{(1)} + c_{21}'\mu_4^{(2)}\mu_4^{(1)} + c_{21}''\mu_6^{(2)}\mu_2^{(1)}$$
$$+ 56D_2\tau_c\mu_6^{(2)} = 0, \tag{A3b}$$

where the general form of the coefficients d_{ij} is given by

$$d_{11}^{(n)} = -1 + p[\gamma_{11}^n + (1-\gamma_{11})^n] + (1-p)[\tilde{\gamma}_{12}]^n, \tag{A4a}$$

$$d_{12}^{(n)} = (1-p)[(1-\tilde{\gamma}_{12})]^n, \tag{A4b}$$

$$d_{22}^{(n)} = -1 + (1-p)[\gamma_{22}^n + (1-\gamma_{22})^n] + p[\tilde{\gamma}_{21}]^n, \tag{A4c}$$

$$d_{21}^{(n)} = p[(1-\tilde{\gamma}_{21})]^n \tag{A4d}$$

and the coefficients a_{ij} are given by

$$a_{11} = 6p[\gamma_{11}(1-\gamma_{11})]^2, \tag{A5a}$$

$$a_{22} = 6(1-p)[\gamma_{22}(1-\gamma_{22})]^2, \tag{A5b}$$

$$a_{12} = 6(1-p)[(1-\tilde{\gamma}_{12})]^2[\tilde{\gamma}_{12}]^2, \tag{A5c}$$

$$a_{21} = 6p[(1-\tilde{\gamma}_{21})]^2[\tilde{\gamma}_{21}]^2. \tag{A5d}$$

Finally, the coefficients b_{ij} are given by

$$b_{11} = 15p\,\gamma_{11}^2(1-\gamma_{11})^2[\gamma_{11}^2 + (1-\gamma_{11})^2], \tag{A6a}$$

$$b_{22} = 15(1-p)\gamma_{22}^2(1-\gamma_{22})^2[\gamma_{22}^2 + (1-\gamma_{22})^2], \tag{A6b}$$

$$b_{12} = 15(1-p)[(1-\tilde{\gamma}_{12})]^4[\tilde{\gamma}_{12}]^2, \tag{A6c}$$

$$b_{12}' = 15(1-p)[(1-\tilde{\gamma}_{12})]^2[\tilde{\gamma}_{12}]^4, \tag{A6d}$$

$$b_{21} = 15p[(1-\tilde{\gamma}_{21})]^2[\tilde{\gamma}_{21}]^4, \tag{A6e}$$

$$b_{21}' = 15p[(1-\tilde{\gamma}_{21})]^4[\tilde{\gamma}_{21}]^2 \tag{A6f}$$

and c_{ij} are

$$c_{11} = 28p\,\gamma_{11}^2(1-\gamma_{11})^2[\gamma_{11}^4 + (1-\gamma_{11})^4], \tag{A7a}$$

$$c_{11}' = 70p\,\gamma_{11}^4(1-\gamma_{11})^4, \tag{A7b}$$

$$c_{12} = 28(1-p)[(1-\tilde{\gamma}_{12})]^6[\tilde{\gamma}_{12}]^2, \tag{A7c}$$

$$c_{12}' = 70(1-p)[(1-\tilde{\gamma}_{12})]^4[\tilde{\gamma}_{12}]^4, \tag{A7d}$$

$$c_{12}'' = 28(1-p)[(1-\tilde{\gamma}_{12})]^2[\tilde{\gamma}_{12}]^6, \tag{A7e}$$

$$c_{22} = 28(1-p)\gamma_{22}^2(1-\gamma_{22})^2[\gamma_{22}^4 + (1-\gamma_{22}^4)], \tag{A7f}$$

$$c_{22}' = 70(1-p)\gamma_{22}^4(1-\gamma_{22})^4, \tag{A7g}$$

$$c_{21} = 28p[(1-\tilde{\gamma}_{21})]^6[\tilde{\gamma}_{21}]^2, \tag{A7h}$$

$$c_{21}' = 70p[(1-\tilde{\gamma}_{21})]^4[\tilde{\gamma}_{21}]^4, \tag{A7i}$$

$$c_{21}'' = 28p[(1-\tilde{\gamma}_{21})]^2[\tilde{\gamma}_{21}]^6. \tag{A7j}$$

It is useful to consider the behavior of the moments in the one-component case. The major simplicity of the resulting formulas allows us to obtain explicit expressions

$$\mu_2 = \frac{D\tau_c}{\Gamma\tau_c + \gamma(1-\gamma)}, \tag{A8a}$$

$$\mu_4 = \frac{12D\tau_c\mu_2 + 6\gamma^2(1-\gamma)^2\mu_2^2}{4\Gamma\tau_c + 1 - \gamma^4 - (1-\gamma)^4}, \tag{A8b}$$

$$\mu_6 = \frac{30D\tau_c\mu_4 + 15\gamma^2(1-\gamma)^2(\gamma^2 + (1-\gamma)^2\mu_2\mu_4}{6\Gamma\tau_c + 1 - \gamma^6 - (1-\gamma)^6}, \tag{A8c}$$

$$\mu_8 = \frac{56D\tau_c\mu_6 + 28\gamma^2(1-\gamma)^2[\gamma^4+(1-\gamma)^4]\mu_2\mu_6 + 70\gamma^4(1-\gamma)^4\mu_4^2}{8\tau_c\Gamma + 1 - \gamma^8 - (1-\gamma)^8}. \tag{A8d}$$

APPENDIX B

Notice that, in order to reduce the number of free parameters, we have assumed the constants, D and Γ, which specify the interactions with the bath to be species independent. Such a choice implies that in the elastic limit both components reach the same granular temperature given by the temperature of the bath, T_b. However, such a requirement could also be satisfied by a different procedure. This second possibility consists of assuming the equation of evolution in the absence of collisions to be given by

$$m_\alpha \partial_t v_i^\alpha(t) = -m_\alpha \kappa v_i^\alpha(t) + \sqrt{2m_\alpha \kappa T_b} W(t), \tag{B1}$$

where $W(t)$ is a Gaussian noise with zero average and variance $\delta(t-t')$.

Thus in the case considered in Sec. II the viscous damping and the stochastic force were the same for both components, in Eq. (B1) the viscous force and the noise variance are proportional to the mass of each species.

Although Eqs. (5) and (B1) admit the same equilibrium distribution functions (6), the relaxation processes are different. In addition, the presence of collisions will generate different results, in particular different temperature ratios.

In fact, the Fokker-Planck for $P_\alpha(v,t)$ for this second case reads

$$\partial_t P_\alpha(v,t) = \kappa \partial_v[v P_\alpha(v,t)] + \frac{\kappa T_b}{m_\alpha} \partial_v^2 P_\alpha(v,t). \tag{B2}$$

For this alternative choice we only give the temperature ratio and comment that it is a monotonically increasing function of ζ in contrast with the behavior illustrated in Sec. III.

$$\frac{T_1}{T_2} = \frac{(1-p)[2\gamma_{22}(1-\gamma_{22})] + p(1-\tilde{\gamma}_{21}^2) + 2\kappa\tau_c + (1-p)\zeta^2(1-\tilde{\gamma}_{12})^2}{p[2\gamma_{11}(1-\gamma_{11})] + (1-p)(1-\tilde{\gamma}_{12}^2) + 2\kappa\tau_c + p\zeta^{-2}(1-\tilde{\gamma}_{21})^2}. \tag{B3}$$

The formula (B3) shows that the dependence on the mass ratio is much weaker than in the case considered in Sec. III. The growth of T_1/T_2 with ζ is similar to that of the homogeneous cooling, although no simple relation exists between the two cases.

Other authors observed the same qualitative increasing trend [23]. However, only the experimental conditions one intends to mimic can resolve the ambiguity relative to the choice of the heat-bath parameters.

[1] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).

[2] K. Feitosa and N. Menon, Phys. Rev. Lett. **88**, 198301 (2002).

[3] A. B. Pippard, *The Elements of Classical Thermodynamics* (Cambridge University Press, Cambridge, 1957).

[4] V. Garzó and J. Dufty, Phys. Rev. E **60**, 5706 (1999).

[5] C. S. Campbell, Annu. Rev. Fluid Mech. **22**, 57 (1990).

[6] M. H. Ernst, Phys. Rep. **78**, 1 (1981).

[7] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **61**, R5 (2000).

[8] P. L. Krapivsky and E. Ben-Naim, J. Phys. A **35**, L147 (2002).

[9] A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Europhys. Lett. **58**, 14 (2002).

[10] A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Phys. Rev. E **65**, 051301 (2002).

[11] U. Marini Bettolo Marconi, A. Baldassarri, and A. Puglisi, Ad. Complex Syst. **4**, 321 (2001).

[12] M. H. Ernst and R. Brito, e-print cond-mat/0112417.

[13] J. A. Carrillo, C. Cercignani, and I. M. Gamba, Phys. Rev. E **62**, 7700 (2000).

[14] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba, J. Stat. Phys. **98**, 743 (2000).

[15] A. Puglisi, V. Loreto, U. Marini Bettolo Marconi, A. Petri, and A. Vulpiani, Phys. Rev. Lett. **81**, 3848 (1998); A. Puglisi, V. Loreto, U. Marini Bettolo Marconi, and A. Vulpiani, Phys. Rev. E **59**, 5582 (1999).

[16] A. V. Bobylev, Sov. Phys. Dokl. **20**, 820 (1976).

[17] U. Marini Bettolo Marconi and A. Puglisi, Phys. Rev. E **65**, 051305 (2002).

[18] T. P. C. van Noije and M. H. Ernst, Granular Matter **1**, 57 (1998).

[19] J. M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).

[20] D. Burnett, Proc. London Math. Soc. **40**, 382 (1935).

[21] G. Peng and T. Ohta, Phys. Rev. E **58**, 4737 (1998).

[22] S. J. Moon, M. D. Shattuck, and J. B. Swift, Phys. Rev. E **64**, 031303 (2001).

[23] A. Barrat and E. Trizac, Granular Matter (to be published), e-print cond-mat/0202297.